Int. J. Heat Mass Transfer. Vol. 3, pp. 126–132. Pergamon Press 1961. Printed in Great Britain.

APPROXIMATE CALCULATION METHOD FOR HEAT TRANSFER IN LAMINAR BOUNDARY LAYERS WITH CONSTANT SURFACE TEMPERATURE

A. G. SMITH* and V. L. SHAH†
Department of Aircraft Propulsion, College of Aeronautics, Cranfield

(*Received* 20 *January* 1961)

Abstract—A simple method for the calculation of heat transfer in a laminar, constant property and constant surface temperature flow, was described by Smith and Spalding [1]. In the present paper, the method is extended to the range of Prandtl numbers 0·7 to 10. Examples are given of heat transfer calculations for ellipses of 2:1 and 4:1 fineness ratio.

Résumé—Une méthode simple de calcul de la transmission de chaleur dans un écoulement laminaire à propriétés constantes et température de surface constante a été décrite par Smith et Spalding [1]. Dans cet article, la méthode est étendue au domaine des nombres de Prandtl compris entre 0,7 et 10. Des exemples de calcul de transmission de chaleur sont donnés pour les ellipses d'excentricité 2:1 et 4:1.

Zusammenfassung—Smith und Spalding [1] beschrieben eine einfache Methode zur Berechnung des Wärmeüberganges in einer Laminarströmung mit gleichbleibenden Stoffeigenschaften und konstanter Temperatur der Berandung. Diese Methode wird hier auf den Bereich der Prandtlzahlen von 0,7 bis 10 ausgedehnt. Als Beispiel wird der Wärmeübergang an Ellipsen vom Achsenverhältnis 2:1 und 4:1 berechnet.

Аннотация—Смит и Сполдинг [1] дали описание простого метода расчета теплообмена в ламинарном потоке с постоянными физическими характеристиками и постоянной температурой поверхности. В публикуемой статье этот метод распространён на область чисел Прандтля от 0,7 до 10. Приводятся примеры расчёта теплообмена для эллипсов с соотношением осей равным 2:1 и 4:1.

NOTATION

A,	number defined in equation (2);
a,	thermal diffusivity, $k/\rho C_p$;
B,	number defined in equation (2);
c,	characteristic length of body, e.g. chord or major axis;
C_p,	specific heat;
Δ_4,	"heat flux thickness" defined by $h = k/\Delta_4$;
E_4,	error term defined in equation (2);
h,	heat transfer coefficient;
k,	conductivity of the fluid;
Nu,	Nusselt number, hc/k;
Nu_x,	Nusselt number, hx/k;
μ,	fluid viscosity;
ν,	kinematic viscosity of the fluid;
Q,	heat;
Re,	Reynolds number, Uc/ν;
Re_x,	Reynolds number, U_1x/ν;
r,	radial dimension of axisymmetric body;
ρ,	density;
σ,	Prandtl number, ν/a;
θ,	temperature;
U_1,	mainstream velocity at a point on the surface;
U,	approach or reference velocity;
u,	fluid velocity within the boundary layer;
X, Y, Z,	vectorial length;
x,	distance along surface from stagnation point;
y,	distance normal to surface.

* Professor of Thermodynamics, University of Nottingham, Nottingham.
† Department of Aircraft Propulsion, College of Aeronautics, Cranfield.

1. INTRODUCTION

EXACT solutions of the boundary layer equations are possible only for a limited range of mainstream velocity distributions. To solve a problem involving the flow of a fluid round a body of arbitrary shape, approximate methods are necessary. Numerous approximate methods are available to calculate heat transfer in a laminar boundary layer with constant fluid properties and constant wall temperature.

Frossling [2] carried out the calculations of the temperature distribution by assuming a power series for mainstream velocity distribution, velocity distribution in the boundary layer and the temperature distribution in the boundary layer, and solved the differential equation. The method is very cumbersome.

Eckert [3] solved the temperature equation by assuming that the rate of growth of the temperature layer is the same as for a wedge flow with the same temperature thickness and the same value of a certain pressure gradient parameter. However, by this method, the calculations for the thermal boundary layer are considerably more tedious than those for the velocity layers.

Squire [4] derived an approximate method by solving an approximate heat flux equation. The Blasius velocity profile was taken, but the displacement thickness was allowed to vary with x in a manner appropriate to the mainstream. The temperature profile was taken similar to the velocity profile but with scale in the y direction altered in a manner to be determined from the energy integral equation.

This method will give good results only for a streamline body because the profiles are assumed to be that of flat plate. Further the method involves more calculation than that of Smith and Spalding [1].

Ambrok [5] has solved an approximate heat flux equation by assuming that a relation of the type $Nu_x = A'(Re_x)^n$ which is true for flat plate, is also valid for a flow over a body of arbitrary shape. This method, though simple, gives lower values of heat transfer when compared to those of Eckert [3].

Allen and Look [6] have applied Reynolds analogy to calculate heat transfer from shear stress for fluids with Prandtl number unity. Frick and McCullough [7] extended this method for any Prandtl number by suggesting a simple multiplier for Prandtl number. This method gives very high values of heat transfer.

The simplest of all the methods appears to be that of Smith and Spalding [1] whose simple quadratures give results identical with those of Eckert [3]. The Smith and Spalding method is limited to fluid with Prandtl number = 0·7 which is adequate for air and some other gases. However, to meet the case of liquids, the method has been extended in the present paper to cover a range of Prandtl numbers up to 10.

2. EXTENSION OF SMITH AND SPALDING METHOD TO A RANGE OF PRANDTL NUMBERS

From vectorial dimensional analysis, it can be shown that the rate of growth of a thermal boundary layer is dependent on the thickness of the layer and on the mainstream velocity gradient, i.e.

$$\frac{U_1}{\nu}\frac{\mathrm{d}(\Delta_4^2)}{\mathrm{d}x} = f\left(\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}, \sigma\right). \tag{1}$$

In deriving the above relation, the following assumptions are made in addition to the boundary layer assumptions:

(1) The rate of growth depends only on local conditions.

(2) Any dependence of the rate of growth on velocity layer shape or thickness can be ignored.

These assumptions lead to

$$\frac{\mathrm{d}\Delta_4}{\mathrm{d}x} = f\left(\Delta_4, U_1, \frac{\mathrm{d}U_1}{\mathrm{d}x}, \mu, k, C_p, \rho\right),$$

and when dimensionless groups are formed allowing a separate identity to the X, Y, Z, length dimensions, equation (1) follows. The dimensions $QT^{-1}\theta^{-1}X^{-1}YZ^{-1}$ and $MT^{-1}X^{-1}YZ^{-1}$ assigned to k and μ respectively, assert the usual boundary layer assumptions that k and μ "act" only through $\partial\theta/\partial y$ and $\partial u/\partial y$. This assertion leads to equation (1) instead of a more general and less useful relation obtained when the three length dimensions are undistinguished.

The unknown function in equation (1) can be obtained from the exact solutions of the thermal boundary layer equations. From exact solution for the thermal boundary layers in "wedge" flows by Eckert [3] (Fig. 1), it can be seen that the relation between the two quantities is nearly linear.

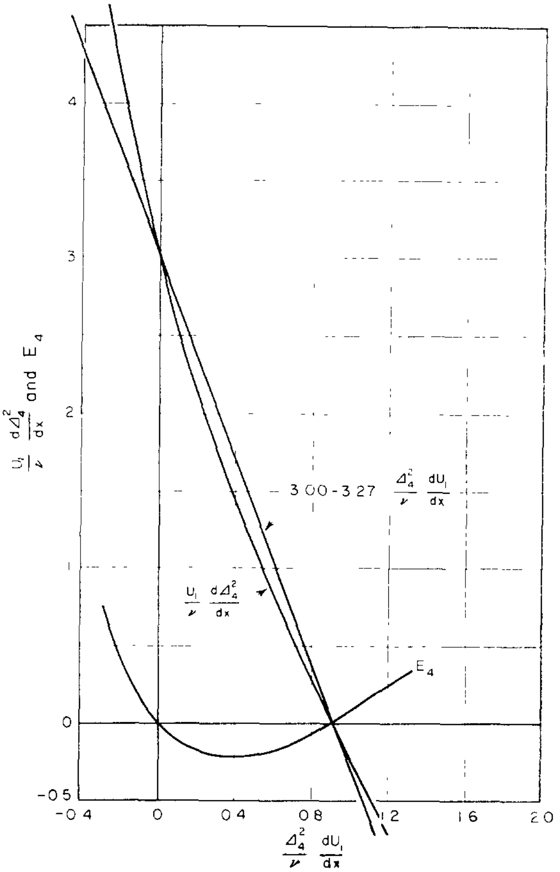

FIG. 1. Typical graph showing relation between $\frac{U_1}{\nu}\frac{d\Delta_4^2}{dx}$ and $\frac{\Delta_4^2}{\nu}\frac{dU_1}{dx}$ for $\sigma = 5$.

Writing the relation

$$\left[\frac{U_1}{\nu}\frac{d\Delta_4^2}{dx}\right] = A - B\left(\frac{\Delta_4^2}{\nu}\frac{dU_1}{dx}\right) + E_4 \qquad (2)$$

where E_4 is the error of the linear approximation, and is a small number dependent on

$$\left(\frac{\Delta_4^2}{\nu}\frac{dU_1}{dx}\right),$$

we may integrate, using the integrating factor U_1^{B-1} to

$$\left[\frac{U_1^B \Delta_4^2}{\nu}\right] = A\int_0^x U_1^{B-1}\,dx + \int_0^x U_1^{B-1} E_4\,dx. \qquad (3)$$

From this quadrature, knowing A and B, Δ_4 may be calculated everywhere, U_1 being a known function of x, and ν being known. The calculation is in principle iterative since E_4 cannot be found until a first approximation to Δ_4 has been obtained. This first approximation, called $\Delta_{4(1)}$ is calculated by omitting the last term from equation (3). From $\Delta_{4(1)}$, $E_{4(1)}$ is calculated, and hence $\Delta_{4(2)}$. Actually $E_{4(1)}$ is often so small that the first approximation $\Delta_{4(1)}$ suffices.

From Δ_4 the heat transfer coefficient may be found by the equation:

$$h = k/\Delta_4. \qquad (4)$$

It is preferable to work with the dimensionless forms of equations (3) and (4). They are:

$$\left(\frac{\Delta_4}{c}\right)^2\left(\frac{Uc}{\nu}\right) = \frac{A}{\left(\frac{U_1}{U}\right)^B}\int_0^{x/c}\left(\frac{U_1}{U}\right)^{B-1} d\left(\frac{x}{c}\right) + \frac{1}{\left(\frac{U_1}{U}\right)^B}\int_0^{x/c}\left(\frac{U_1}{U}\right)^{B-1} E_4\, d\left(\frac{x}{c}\right) \qquad (5)$$

and

$$\frac{hc}{k}\Big/\sqrt{\left(\frac{Uc}{\nu}\right)} = 1\Big/\left(\frac{\Delta_4}{c}\right)\sqrt{\left(\frac{Uc}{\nu}\right)} \qquad (6)$$

E_4 is now dependent on

$$\left(\frac{\Delta_4}{c}\right)^2\left(\frac{Uc}{\nu}\right)\frac{d(U_1/U)}{d(x/c)}.$$

At the front stagnation point, equation (5) becomes

$$\left(\frac{\Delta_4}{c}\right)^2\left(\frac{Uc}{\nu}\right) = \frac{A/B}{\frac{d(U_1/U)}{d(x/c)}}. \qquad (7)$$

Eckert [3] gives data sufficing for the calculation of A, B, and E_4 for $\sigma = 0{\cdot}7$, 0·8, 1·0, 5·0 and 10. Smith and Spalding [1] gave A, B, for $\sigma = 0{\cdot}7$ and the values for other σ are added below, in Table 1.

Table 1

σ	0·7	0·8	1·0	5	10
A	11·68	10·61	9·07	3·00	1·885
B	2·87	2·90	2·95	3·27	3·41
A/B	4·07	3·66	3·07	0·919	0·554

Figure 1 shows the relation between

$$\frac{U_1}{\nu}\frac{\mathrm{d}(\Delta_4^2)}{\mathrm{d}x} \text{ and } \frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}$$

for $\sigma = 5$, as an example. Similar graphs were drawn for other σ values to obtain the A and B values of Table 1.

E_4 is given in Table 2.

Tables 1 and 2 give all the numerical data necessary for computing heat transfer in laminar layers over the range $\sigma = 0{\cdot}7$ to 10. For convenience in using the method at intermediate σ values, the dependence of A and B on σ has been plotted in Fig. 2.

Equations (5) and (6) have been computed by the authors, for flow over ellipses of major/minor axis ratio 4:1 and 2:1, and for Prandtl numbers 0·7, 0·8, 1·0, 5·0 and 10. Integration was performed by Simpson's rule using 0·01 intervals to $x/c = 0{\cdot}1$, then 0·05 intervals to $x/c = 0{\cdot}7$. An extract from the computation for $\sigma = 5$, for the 4:1 ellipse, is given in Table 3. The results of these calculations are shown in Figs. 3 and 4, and the velocity distributions which were used are shown in Fig. 5.

Comparison of these calculations with other theoretical results is not possible except at $\sigma = 0{\cdot}7$, where results from Eckert [3] and Schuh [8] are available. These results are in too close agreement with those of the present paper for differences to appear when plotting to the scale of Figs. 3 and 4.

3. AXISYMMETRIC FLOW

By the use of Mangler's transformations [9]:

(1) $$\bar{x} = \int_0^x \left(\frac{r}{c}\right)^2 \mathrm{d}x$$

(2) $$\bar{y} = \left(\frac{r}{c}\right) y$$

(3) $$\bar{\Delta} = \left(\frac{r}{c}\right) \Delta$$

Table 2

$\sigma = 0{\cdot}7$	$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}$	6·05	4·07	2·26	1·01	0	−1·02
	E_4	1·042	0	−0·68	−0·67	0	2·06
$\sigma = 0{\cdot}8$	$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}$	5·427	3·663	2·039	0·917	0	−0·937
	E_4	1·027	0·011	−0·618	−0·614	0	1·935
$\sigma = 1{\cdot}0$	$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}$	4·541	3·074	1·721	0·779	0	−0·809
	E_4	0·919	−0·002	−0·551	−0·543	0	1·716
$\sigma = 5$	$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}$	1·327	0·919	0·531	0·248	0	−0·287
	E_4	0·343	0	−0·202	−0·208	0	0·740
$\sigma = 10$	$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}U_1}{\mathrm{d}x}$	0·798	0·554	0·325	0·154	0	−0·185
	E_4	0·238	0	−0·126	−0·132	0	0·491

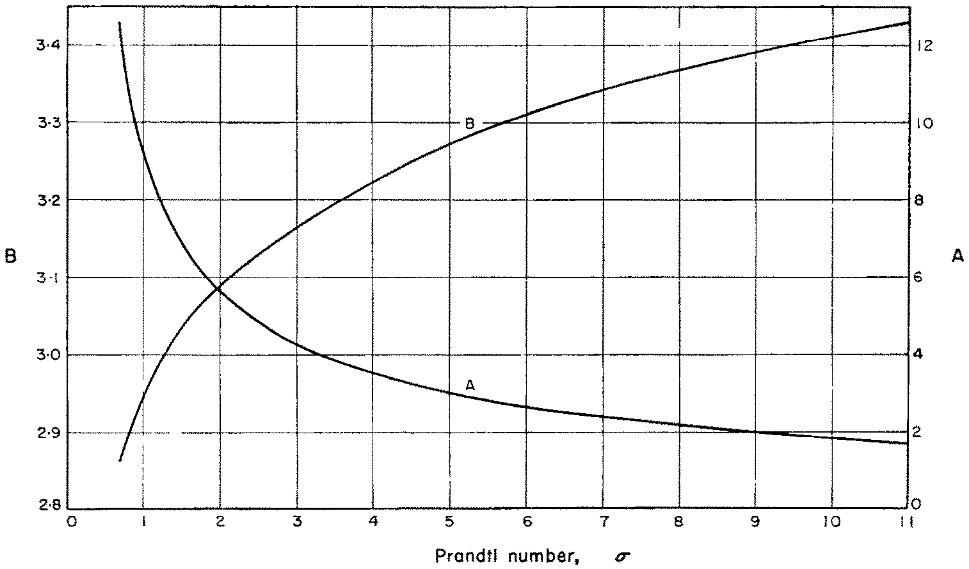

FIG. 2. Variation of A and B in equation (2) with Prandtl number.

Table 3

x/c	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
U_1/U	0	1·153	1·22	1·239	1·246	1·25	1·249	1·245	1·235
$I = \int_0^{x/c} (U_1/U)^{2\cdot 27} \, d\left(\frac{x}{c}\right)$	0	0·087	0·237	0·397	0·561	0·726	0·892	1·057	1·22
$\left(\frac{\Delta_4}{c}\right)^2 \left(\frac{Uc}{\nu}\right) = \frac{3\cdot 00\, I}{(U_1/U)^{3\cdot 27}}$	0·023 *	0·164	0·371	0·591	0·821	1·295	1·295	1·55	1·839
$\frac{hc/k}{\sqrt{(Uc/\nu)}}$ 1st approx.	6·601	2·47	1·64	1·3	1·105	0·976	0·88	0·803	0·737
$\left(\frac{\Delta_4}{c}\right)^2 \left(\frac{Uc}{\nu}\right) \frac{d(U_1/U)}{d(x/c)}$	0·919	0·232	0·113	0·07	0·041	0·011	−0·027	−0·098	−0·277
E_4	0	−0·204	−0·144	−0·105	−0·07	−0·02	−0·043	0·185	0·685
$\int_0^{x/c} (U_1/U)^{2\cdot 27} E_4 \, d\left(\frac{x}{c}\right)$	0	−0·018	−0·044	−0·063	−0·078	−0·085	−0·084	−0·077	0·004
$\frac{1}{(U_1/U)^{3\cdot 27}} \int_0^{x/c} \left(\frac{U_1}{U}\right)^{2\cdot 27} E_4 \, d\left(\frac{x}{c}\right)$	0	−0·011	−0·023	−0·031	−0·038	−0·041	−0·04	−0·038	−0·002
$\frac{hc}{k} \Big/ \sqrt{\left(\frac{Uc}{\nu}\right)}$ 2nd approx.	6·601	2·559	1·694	1·336	1·13	0·995	0·893	0·813	0·738

No further cycles are needed.

* From the front stagnation point relation.

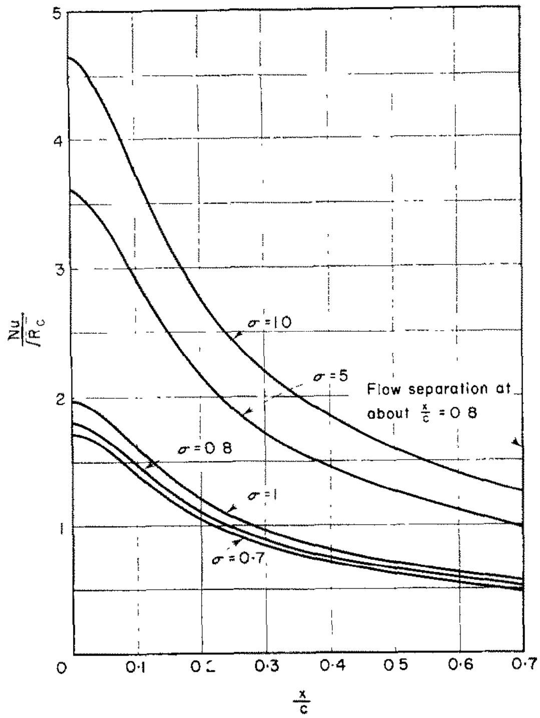

FIG. 3. Distribution of Nusselt number of the surface of a 2:1 ellipse for various Prandtl numbers.

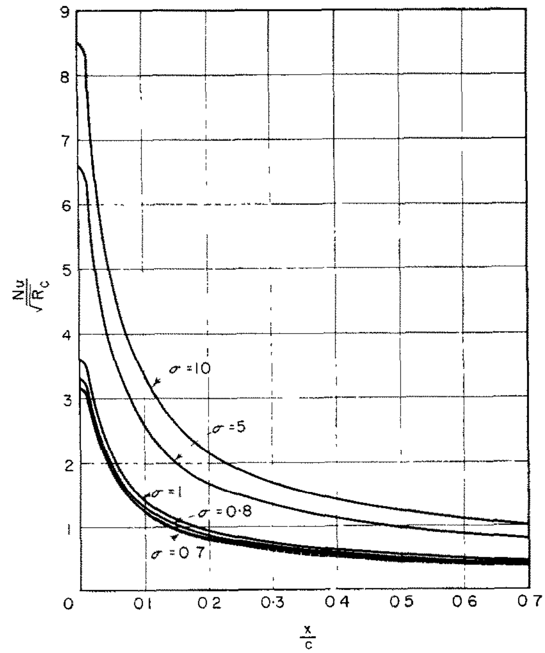

FIG. 4. Distribution of Nusselt number on the surface of a 4:1 ellipse for various Prandtl numbers.

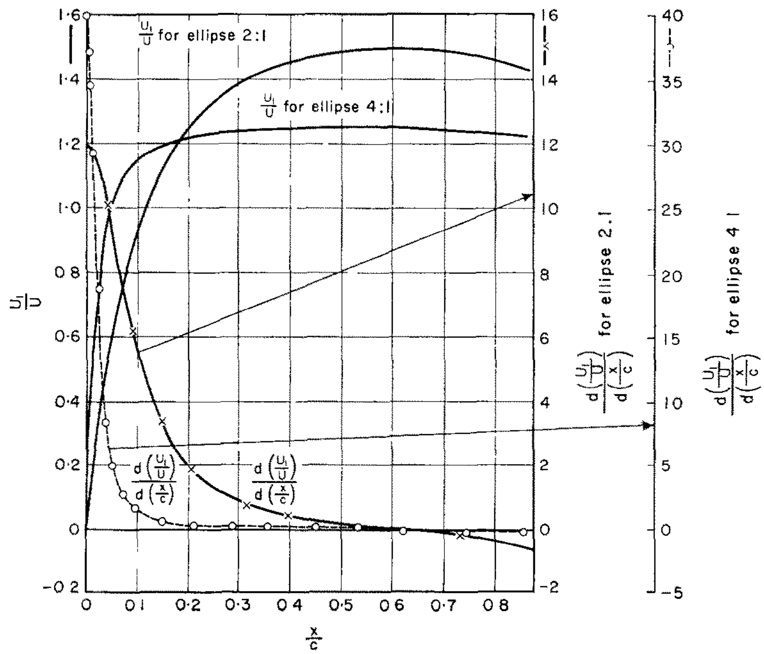

FIG. 5. Velocity distributions on the 2:1 and 4:1 ellipses.

(4) $\bar{h} = (c/r)\, h$

(5) $\bar{U}_1 = U_1.$

the method given above can be used for axi-symmetric flows. Given $U_1(x)$ and $r(x)$, $\bar{U}_1(\bar{x})$ may be tabulated and $\bar{h}$ determined from equations (5) and (6) and hence h.

4. CONCLUSION

The method described and illustrated in detail above, gives a very rapid calculation of heat transfer in the two-dimensional or axi-symmetric cases to which it can be applied. It is difficult to assess the accuracy of the calculations: for "wedge" flows the results should be exact, as should those of Eckert. The basis of the present method is in fact simply that on a surface with an arbitrary U_1 distribution, the rate of growth of thermal layer thickness is the same as it would be in a wedge flow with the same thermal thickness, U_1 and dU_1/dx. The present method is very unlikely to give highly misleading predictions of heat transfer: stagnation point solutions are exact within the usual boundary layer assumptions, and often large portions of subsequent velocity distributions are similar to "wedge" flow distributions. The method should be used with some regard for the velocity-layer phenomena—it would be fortuitous if the method gave accurate results in a region of flow separation, but the authors believe that it can be used with some confidence to near the calculated separation point.

REFERENCES

1. A. G. Smith and D. B. Spalding, Heat transfer in a laminar boundary layer with constant fluid properties and constant wall temperature. *J. Roy. Aero. Soc.* **62**, 60 (1958).
2. N. Frossling, Verdunstung, Wärmeübergang und Geschwindigkeitsverteilung bei zweidimensionaler und rotationssymmetrischer laminarer Grenzschichtströmung. *N.F. Avd.* **2**, 36 (1940).
3. E. R. G. Eckert, Die Berechnung des Warmeübergangs in der laminaren Grenzschicht umströmter Körper. *Z. Ver. Dtsch. Ing. Forschungsheft* 416 (1942).
4. H. B. Squire, Heat transfer calculations for aerofoils. *Aero. Res. Coun. Lond. R. and M.* 1986 (1942).
5. G. S. Ambrok, Approximate solution of equations for the thermal boundary layer with variation of boundary layer structure. *Soviet Fiz. Tech. Fiz.* **2**, 1979 (1957).
6. H. J. Allen and B. C. Look, A method for calculating heat transfer in the laminar flow region of bodies. *N.A.C.A. Rep.* 764 (1943).
7. C. W. Frick, Jr. and G. B. McCullough, A method for determining the rate of heat transfer from a wing or streamline body. *N.A.C.A. Rep.* 830 (1945).
8. H. Schuh, A new method for calculating laminar heat transfer on cylinders of arbitrary cross-section and on bodies of revolution at constant and variable wall temperature. *K.T.H. Aero T.N.* 33, Stockholm (1953.)
9. W. Mangler, Zusammenhang zwischen ebenen und rotationssymmetrischen Grenzschichten in Kompressiblen Flussigkeiten. *Z. Angew. Math. Mech.* 28 (1948).